APPROXIMATIONS OF THE EIGENVALUES OF THE COVARIANCE
MATRIX OF A FIRST-ORDER AUTOREGRESSIVE PROCESS

R.J. STROEKER*
Erasmus University, 3000 DR Rotterdam, The Netherlands

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This paper provides explicit estimates of the eigenvalues of the covariance matrix of an
autoregressive process of order one. Also explicit error bounds are established in closed form.
Typically, such an error bound is given by 
\[ e_k = \left(\frac{4}{n+1}\right)\rho^k \sin(k\pi/(n+1)), \]
so that the approximations improve as the size of the matrix increases. In other words, the accuracy of the
approximations increases as direct computations become more costly.

1. Introduction

Consider the autoregressive process of order one,
\[ Y_t = \rho Y_{t-1} + e_t, \quad t \in \mathbb{Z}, \]
where \( \rho \) is real with \( |\rho| < 1 \), \( e_t \) is stochastically uncorrelated with the other \( e \)'s
and with all \( Y_s \) for \( s < t \). Moreover, \( e_t \) has zero mean and variance \( \sigma^2 \). The
time series \( Y_t \) converges as \( t \to \infty \) to a stationary time series, because \( |\rho| < 1 \).
The \( n \times n \) covariance matrix of \( n \) observations has the form \( \sigma^2 (1 - \rho^k)^{-1} \Gamma \),
where \( \Gamma = \Gamma_{n,\rho} \) is the matrix with \( k \)th row vector
\[ (\rho^k, \rho^{k-1}, \ldots, \rho, 1, \rho, \ldots, \rho^{n-k-1}, \rho^{n-k}), \quad k = 1, \ldots, n. \]

Sometimes, e.g. in connection with power comparisons of a test of the hypothesis \( \rho = \rho_0 \) [cf. Dickey and Fuller (1979)] or in the spectral theory of
time series [cf. Fuller (1976, ch. 4)], it might be useful to have information on
the eigenvalues of \( \Gamma \) in explicit form. However, there appears to be no
explicit form for the eigenvalues [see Grenander and Szegö (1958, p. 70)],
except for the trivial case \( \rho = 0 \), which we shall exclude from now on. In this
paper we intend to give certain approximations of the eigenvalues of \( \Gamma \),
which are especially useful in case \( n \) is large; for small \( n \) an easy algorithm is
available for exactly calculating the eigenvalues.

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In order to obtain information on the eigenvalues of $\Gamma$, we consider the tridiagonal $n \times n$ matrix

\[
(1 - \rho^2)\Gamma^{-1} = \begin{pmatrix}
1 & -\rho & 0 & \cdots & 0 \\
-\rho & 1 + \rho^2 & -\rho & \cdots & 0 \\
0 & -\rho & 1 + \rho^2 & -\rho & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -\rho & 1
\end{pmatrix}.
\] (1)

It seems reasonable to compare the eigenvalues $\mu_k$ of $(1 - \rho^2)\Gamma^{-1}$ with those of a slightly perturbed matrix, chosen in such a way that its eigenvalues can be given explicitly. With this in mind, we consider the $n \times n$ matrices

\[ M_{\text{sign}(\delta)} = M_{n, \rho}(\delta) := (1 - \rho^2)\Gamma^{-1} + \rho(\rho + \delta)E, \] (2)

where $\delta = -1, 0$ or $1$, and $E := \text{diag}(1, 0, \ldots, 0, 1)$. The eigenvalues $\nu_k^{\text{sign}(\delta)}$ and corresponding eigenvectors of $M_{\text{sign}(\delta)}$ can be given explicitly, as we shall see in section 2. It is reasonable to expect that the Rayleigh Quotient of the normalized eigenvector $x_k$ corresponding to $\nu_k^{\text{sign}(\delta)}$ with respect to the matrix $(1 - \rho^2)\Gamma^{-1}$ [cf. Wilkinson (1965, ch. 3, §54)], i.e.,

\[ \zeta_k := x_k'(1 - \rho^2)\Gamma^{-1}x_k, \]

provides a better approximation to $\mu_k$ than $\nu_k^{\text{sign}(\delta)}$ does. Hence we are led to the following approximations of the eigenvalues $\mu_k$ of $(1 - \rho^2)\Gamma^{-1}$ (the $\mu_k$'s are arranged in increasing order as $\rho > 0$ and in decreasing order as $\rho < 0$):

\[
\zeta_k^+ = (1 - \rho^2) + \frac{2}{n} \rho(1 - \rho), \quad k = 1, \\
= 1 - 2\rho \cos \left(\frac{(k-1)\pi}{n}\right) + \rho^2 + \frac{2}{n} \rho(1 - \rho) \left(1 + \cos \left(\frac{(k-1)\pi}{n}\right)\right), \quad k = 2, \ldots, n, \\
\zeta_k^+ = 1 - 2\rho \cos \left(\frac{k\pi}{n+1}\right) + \rho^2 + \frac{4}{n+1} \rho^2 \left(\sin \left(\frac{k\pi}{n+1}\right)\right)^2, \quad k = 1, \ldots, n, \quad (3)
\]

\[
\zeta_k^- = 1 - 2\rho \cos \left(\frac{k\pi}{n}\right) + \rho^2 - \frac{2}{n} \rho(1 + \rho) \left(1 - \cos \left(\frac{k\pi}{n}\right)\right), \quad k = 1, \ldots, n - 1, \\
= (1 + \rho)^2 - \frac{2}{n} \rho(1 + \rho), \quad k = n.
\]
The corresponding error bounds for \( \mu_k \) are given by

\[
\varepsilon_k^- = \left( \frac{2}{n} \right)^{\frac{k}{2}} (1 - \rho), \quad k = 1,
\]

\[
= \left( \frac{4}{n+1} \right)^{\frac{k}{2}} \rho (1 - \rho) \cos \frac{(k-1)\pi}{2n}, \quad k = 2, \ldots, n,
\]

\[
\varepsilon_k^+ = \left( \frac{4}{n+1} \right)^{\frac{k}{2}} \rho^2 \sin \frac{k\pi}{n+1}, \quad k = 1, \ldots, n.
\]

(4)

\[
\varepsilon_k^+ = \left( \frac{4}{n} \right)^{\frac{k}{2}} \rho (1 + \rho) \sin \frac{k\pi}{2n}, \quad k = 1, \ldots, n-1,
\]

\[
= \left( \frac{2}{n} \right)^{\frac{k}{2}} \rho (1 + \rho), \quad k = n.
\]

Because \( \lambda_k = (1 - \rho^2)/\mu_k \), the error bounds for \( \lambda_k \) are accordingly given by expressions like \( (1 - \rho^2)\varepsilon_k^/\varepsilon_k^2 \), where if necessary + or – superscripts are added.

These error bounds show that for increasing \( n \) the approximations improve. For small values of \( n \), it is not difficult to find good estimates by direct calculation, as we shall see in the next section.

2. Exact evaluation of eigenvalues

In this section we present an algorithm for the exact computation of the eigenvalues \( \mu_k \) of \( (1 - \rho^2)\Gamma^{-1} \) and thus also of the eigenvalues \( \lambda_k \) of \( \Gamma \), because \( \lambda_k = (1 - \rho^2)/\mu_k \). The algorithm is based on well-known facts, most of which are easily accessible in the literature [cf. Grenander and Szegö (1958, ch. 5)]. Nevertheless we shall provide short proofs because they have a bearing on the arguments used in the next section.

Assume that \( \mu \) is an eigenvalue of \( (1 - \rho^2)\Gamma^{-1} \). We introduce \( \alpha \) by putting \( \mu = 1 + 2\rho\alpha + \rho^2 \). Then

\[
0 = \det(\mu I - (1 - \rho^2)\Gamma^{-1})
\]

\[
= \begin{vmatrix}
2\rho\alpha + \rho^2 & \rho & 0 & \ldots & 0 \\
\rho & 2\rho\alpha & \rho & \ldots & 0 \\
0 & \rho & 2\rho\alpha & \rho & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & \ldots & 0 & \rho & 2\rho\alpha + \rho^2
\end{vmatrix} =: \rho^n A_n(\alpha).
\]
Consequently, the values of $\mu$ are in one-to-one correspondence with the roots of the equation $A_n(x) = 0$ in the variable $x$.

In order to obtain a manageable expression for $A_n(x)$, we proceed as follows. For each $n \in \mathbb{N}$, let $U_n(x)$ be the unique $n$th degree Chebyshev polynomial of the second kind, normalized by $U_n(1) = n + 1$. It is a well-known fact [for general information on the Chebyshev polynomials, see Abramowitz and Stegun (1965) and Rivlin (1974)] that $U_n(x)$ may be written as

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & \ldots & \ldots & 0 \\ 1 & 2x & 1 & \ldots & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots & \ldots & 0 \\ \vdots & \vdots & \ldots & \ddots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 1 & 2x \end{vmatrix}.$$ 

By direct evaluation of the defining determinant for $A_n(x)$, we find that

$$A_n(x) = U_{n-2}(x)\rho^2 + 2U_{n-1}(x)\rho + U_n(x),$$

identically in $\rho$ and $x$. In order to locate the roots of $A_n(x) = 0$, we consider the zero's of the Chebyshev polynomials $U_n(x)$. We define

$$\zeta_k^{(n+1)} := \cos \frac{k\pi}{n+1} \quad \text{for} \quad k = 0, 1, \ldots, n+1 \quad \text{and} \quad n \in \mathbb{N}.$$ 

Since $U_n(x) = \sin (n+1)\theta / \sin \theta$ if $x = \cos \theta \ ( -1 < x < 1)$, it follows that

$$U_n(\zeta_k^{(n+1)}) = 0 \quad \text{for} \quad k = 1, \ldots, n.$$ 

Now we are in a position to prove:

**Proposition 1.** The polynomial $A_n$ of degree $n$ has exactly $n$ simple zero's $\alpha_1, \ldots, \alpha_n$ in the interval $(-1, 1)$. If $-1 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < 1$, then for $k = 1, \ldots, n$ we have

$$\alpha_k \in I_k^+ := (\frac{\zeta(n)}{\zeta(n-k+1)}, \frac{\zeta(n+1)}{\zeta(n-k+1)}) \quad \text{if} \quad 0 < \rho < 1,$$

$$\alpha_k \in I_k^- := (\frac{\zeta(n+1)}{\zeta(n-k+1)}, \frac{\zeta(n)}{\zeta(n-k)}) \quad \text{if} \quad -1 < \rho < 0.$$ 

Moreover,

$$\lim_{\rho \to 1^{-1}} \alpha_k = \frac{\zeta(n)}{\zeta(n-k)}, \quad \lim_{\rho \to -1} \alpha_k = \frac{\zeta(n+1)}{\zeta(n-k+1)}, \quad \lim_{\rho \to 0^+} \alpha_k = \frac{\zeta(n)}{\zeta(n-k+1)}, \quad \lim_{\rho \to 1} \alpha_k = \frac{\zeta(n+1)}{\zeta(n-k+1)}. $$
Proof. The Chebyshev polynomials $U_n(x)$ satisfy the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

Since

$$A_n(x) = U_{n-2}(x)p^2 + 2U_{n-1}(x)p + U_n(x), \quad U_{n-1}(\xi^{(n)}_k) = 0 \quad \text{for} \quad k = 1, \ldots, n-1,$$

we find that for these values of $k$,

$$A_n(\xi^{(n)}_k) = (-1)^k (1 - \rho^2),$$

$$A_n(1) = n(1 + \rho)^2 + 1 - \rho^2,$$

$$A_n(-1) = (-1)^n [n(1 - \rho)^2 + 1 - \rho^2].$$

Consequently,

$$A_n(\xi^{(n)}_k), A_n(\xi^{(n)}_{k-1}) < 0 \quad \text{for} \quad k = 1, \ldots, n.$$

This shows that $A_n$ changes sign at least $n$ times on the interval $(-1,1)$. Because $A_n$ is a polynomial of degree $n$, this means that $A_n$ has exactly $n$ simple zeros in $(-1,1)$. Moreover,

$$\alpha_{n-k+1} \in J_k := (\xi^{(n)}_k, \xi^{(n)}_{k-1}) \quad \text{for} \quad k = 1, \ldots, n.$$

Because $(k-1)/n < k/(n+1) < k/n$ for $k = 1, \ldots, n$, we see that

$$\xi^{(n+1)}_k \in J_k.$$

Furthermore,

$$A_n(\xi^{(n+1)}_k) = 2\rho(-1)^{k-1}(1 + \rho\xi^{(n+1)}_k).$$

Hence

$$A_n(\xi^{(n+1)}_k) : A_n(\xi^{(n)}_k) = -2\rho(1 - \rho^2)(1 + \rho\xi^{(n+1)}_k)$$

and $\rho$ have opposite signs. This proves that

$$\alpha_k \in I^+_k, \quad 0 < \rho < 1 \quad \text{and} \quad \alpha_k \in I^-_k, \quad -1 < \rho < 0.$$

Clearly, $\alpha_k$ is a continuous function of $\rho$. Using the defining recurrence relation for the Chebyshev polynomials, the limits $\lim_{\rho \to \delta} \alpha_k$ are now easily calculated. Q.E.D.
These results make it possible to construct a simple algorithm for the direct computation of the $\mu$'s. Indeed, this proposition gives a set of $n$ disjoint intervals $I_k^+$ (or $I_k^-$) with the following properties:

(i) each interval contains precisely one (simple) zero of $\Delta_n$, and
(ii) the signs of the values of $\Delta_n$ at the end points of each interval are opposite.

Thus the rule of False Position (Regula Falsi) is suitable for application in each interval. In order to avoid slow convergence, which may happen when an interval is reached on which the function $\Delta_n$ is convex or concave, a refinement of the Regula Falsi method can be used, for instance the so-called Illinois algorithm [see Rabinowitz (1970, p. 25)].

We programmed the TI-59 pocket calculator accordingly and we found that for given $n$, $\rho$ and precision $\eta = 10^{-10}$, it took approximately one minute computation time to calculate each $\mu$, independent of the size of the matrix $\Gamma$.

3. Approximations and error bounds

In order to approximate the eigenvalues of the matrix $(1-\rho^2)\Gamma^{-1}$, we need the eigenvalues and corresponding eigenvectors of the matrices $M_{\text{sign}(\delta)} = (1-\rho^2)\Gamma^{-1} + \rho(\rho + \delta)I$ explicitly.

Let $v = 1 + 2\rho\beta + \rho^2$ be an eigenvalue of $M_{\text{sign}(\delta)}$. As in the previous section, we deduce that

$U_\delta(\beta) = 0, \quad \delta = 0 \quad \text{and} \quad (\beta + \delta)U_{\delta-1}(\beta) = 0, \quad \delta = +1.$

This yields:

Proposition 2. The matrix $M_{\text{sign}(\delta)}$ has eigenvalues $v_k^{\text{sign}(\delta)}$, $k = 1, \ldots, n$, with corresponding orthogonal eigenvectors $x_k^{\text{sign}(\delta)}$ with components $x_{jk}^{\text{sign}(\delta)}$, $j = 1, \ldots, n$:

$\delta = -1$

$v_k^- = 1 + 2\rho\zeta^{(n)}_{n-k+1} + \rho^2 = 1 - 2\rho \cos \frac{(k-1)\pi}{n} + \rho^2$,

$x_{jk}^- = \cos \frac{(2j-1)(k-1)\pi}{2n}$,

$\|x_1^+\|_2 = n^{\frac{1}{4}}, \quad \|x_k^+\|_2 = (\frac{1}{2}n)^{\frac{1}{4}}$ for $k \neq 1.$
\[ \delta = 0 \]

\[ v_k = 1 + 2\rho \zeta_n^{(n+1)} + \rho^2 = 1 - 2\rho \cos \frac{k\pi}{n+1} + \rho^2, \]

\[ x_{jk} = \sin \frac{kj\pi}{n+1}, \]

\[ \|x_k\|_2 = \left(\frac{1}{2}(n+1)\right)^{\frac{1}{2}}. \]

\[ \delta = 1 \]

\[ v_k^+ = 1 + 2\rho \zeta_n^{(n-1)} + \rho^2 = 1 - 2\rho \cos \frac{k\pi}{n+1} + \rho^2, \]

\[ x_{jk}^+ = \sin \frac{(2j-1)k\pi}{2n}, \]

\[ \|x_k^+\|_2 = \left(\frac{1}{2}n\right)^{\frac{1}{2}} \text{ for } k \neq n, \quad \|x_n^+\|_2 = n^{\frac{1}{2}}. \]

We omit the derivation of the eigenvectors. Although somewhat tedious, the calculations involved are elementary. Note that \( M (\delta = 0) \) is the \( n \times n \) covariance matrix of a moving average process of order one and that \( v_k \) given in Proposition 2 agrees with the value given in Grenander and Szegö (1958, p. 67).

It is not difficult to see that the cases \( \delta = 1 \) and \( \delta = -1 \) are very similar. Indeed, changing \( \rho \) into \( -\rho \) and \( k \) into \( n-k+1 \) has the effect of transforming \( \delta = 1 \) into \( \delta = -1 \) and vice versa.

From now on we shall, for obvious reasons of simplicity, drop the sign of \( \delta \) whenever there is no risk of confusion. So \( v_k \) can also mean \( v_k^+ \) or \( v_k^- \), etc.

Using \( v_k \) as a first approximation of \( \mu_k \), standard perturbation theory indicates that the Rayleigh Quotient of \( x_k \) with respect to the matrix \((1-\rho^2)\Gamma^{-1}\), i.e.,

\[ \zeta_k = \frac{x_k(1-\rho^2)\Gamma^{-1}x_k}{\|x_k\|^2}, \]

is likely to give a better approximation to the eigenvalue \( \mu_k \) of \((1-\rho^2)\Gamma^{-1}\) than \( v_k \) (the corresponding indices of \( \zeta_k, \mu_k \) and \( v_k \) indicate that their orderings agree); see Wilkinson (1965, ch. 3, §54, p. 172). Clearly,

\[ \zeta_k = v_k - \rho(\rho + \delta)(x_{1k}^2 + x_{nk}^2)/\|x_k\|^2, \]

and this expression directly leads to (3).
To establish the error bounds in (4) we proceed as follows.

From now on we assume \( \rho \) to be positive. This gives no loss of generality, because changing \( \rho \) into \(-\rho\) can be effected by changing the sign of \( \delta \) and reversing the order of magnitude of the eigenvalues, e.g., \( \mu_k^+(-\rho) = \mu_{n-k+1}^-(-\rho) \), \( \zeta_k^+(-\rho) = \zeta_{n-k+1}^-(\rho) \) and also \( \zeta_k^-(\rho) = \zeta_{n-k+1}^+(-\rho) \). Since we have assumed \( \rho \) to be positive \((0 < \rho < 1)\), the eigenvalues \( \mu_k \) of \( M \) as given in Proposition 2 are arranged in increasing order. If we agree to a similar ordering of the \( \mu \)'s \((\mu_1 < \mu_2 < \ldots < \mu_n)\), then by the Courant–Fischer theorem [cf. Ortega (1972, 3.3.2, p. 57)] we have

\[
|v_k - \mu_k| \leq \|\rho(\rho + \delta)E\|_2 = \rho(\rho + \delta) \quad \text{for} \quad k = 1, \ldots, n.
\]

Moreover, it is an easy consequence of the Wielandt–Hoffman theorem [cf. Wilkinson (1965, ch. 2 §48, p. 104)] that

\[
\min_k |v_k - \mu_k| \leq \left(\frac{2}{n}\right)^{\frac{1}{2}} \rho(\rho + \delta).
\]

This gives approximately the right upper bound for all absolute differences \( |v_k - \mu_k| \) as we shall see presently. Define

\[
\varepsilon_k := \|(1 - \rho^2)\Gamma^{-1} - v_k I_n\|_2 \left\| x_k \right\|_2.
\]

This way we obtain formulae (4).

From elementary linear algebra we use the following result:

Let \( A \) be a symmetric \( n \times n \) matrix and let \( x \in \mathbb{R}^n \) be a given vector normalized by \( \left\| x \right\|_2 = 1 \). Then for every \( \lambda_0 \in \mathbb{R} \),

\[
\min_{\lambda} |\lambda - \lambda_0| \leq \left\| Ax - \lambda_0 x \right\|_2, \tag{\text{a}}
\]

where \( \lambda \) runs through all eigenvalues of \( A \). Moreover, \( \left\| Ax - \lambda x \right\|_2 \) is minimized by \( \lambda = x^\top A x \).

A proof may be found in, e.g., Fox (1964, p. 279).

Application of this result yields

\[
\min_i |\mu_i - v_k| \leq \varepsilon_k,
\]

and thus

\[
|\mu_k - v_k| \leq \varepsilon_k \quad \text{and} \quad |\mu_k - \zeta_k| \leq \varepsilon_k,
\]

provided \( \varepsilon_k \) is small enough.
Clearly, to proceed from here, we need additional information on the separation of the eigenvalues \( \mu_k \) of \((1 - \rho^2) \Gamma^{-1}\). Or, as we do not know the \( \mu \)'s, we could try the computable quantities defined by

\[
\eta_k := \min_{i \neq k} |\zeta_i - \zeta_k|.
\]

Indeed, \( \eta_k > 2\varepsilon_k \) for all \( k \), implies that the \( \varepsilon_k \)-neighbourhoods of the \( \zeta_k \)'s are mutually disjoint and hence

\[
|\mu_k - \zeta_k| \leq \varepsilon_k \quad \text{for all} \quad k.
\] (5)

To improve upon this estimate, we assume that \( \eta_k > 2\varepsilon_k \) for all \( k \). Consequently, if \( \varepsilon \geq \max_k \varepsilon_k \) and \( \min_k \eta_k > 2\varepsilon \), we have

\[
\min_{i \neq k} |\mu_i - \zeta_k| \geq \eta_k - \varepsilon > \varepsilon.
\]

Now let \( \{y_1, \ldots, y_n\} \) be an orthonormal basis of \( \mathbb{R}^n \) consisting of eigenvectors of \((1 - \rho^2) \Gamma^{-1}\), so that

\[
(1 - \rho^2) \Gamma^{-1} y_k = \mu_k y_k.
\]

Put

\[
x_k/\|x_k\|_2 = \sum_{i=1}^n \zeta_{ik} y_i \quad \text{for all} \quad k.
\]

Then

\[
\sum_{i=1}^n \zeta_{ik}^2 = 1 \quad \text{and} \quad \zeta_k = \sum_{i=1}^n \mu_i \zeta_{ik}^2.
\]

Now

\[
\varepsilon_k^2 \geq \|\{(1 - \rho^2) \Gamma^{-1} - \zeta_k I_n\} x_k/\|x_k\|_2\|_2^2 = \sum_{i=1}^n (\mu_i - \zeta_k^2) \zeta_{ik}^2
\]

\[
\geq (\eta_k - \varepsilon) \cdot \sum_{i=1, i \neq k}^n |\mu_i - \zeta_k| \zeta_{ik}^2 \geq (\eta_k - \varepsilon) |\mu_k - \zeta_k| \zeta_{kk}^2.
\]

Hence, for \( k = 1, \ldots, n \), we have \( |\mu_k - \zeta_k| \leq \varepsilon_k^2 / (\eta_k - \varepsilon) \zeta_{kk}^2 \) and this gives

\[
|\mu_k - \zeta_k| \leq \varepsilon_k^2 / (\eta_k - 2\varepsilon),
\] (6)
because

\[ \lambda_{kk}^2 = 1 - \sum_{i=1, i \neq k}^{n} \lambda_{ik}^2 \geq 1 - (\eta_k - \varepsilon)^{-2} \sum_{i=1}^{n} (\lambda_i - \zeta_k)^2 \lambda_{ik}^2 \]

\[ \geq 1 - (\eta_k - \varepsilon)^{-2} \zeta_k^2 \geq 1 - (\eta_k - \varepsilon)^{-2} \varepsilon^2 > (\eta_k - 2\varepsilon)/(\eta_k - \varepsilon). \]

Obviously, if \( \eta_k \gg \varepsilon \), then (6) gives a better upper bound than (5). From the expression for \( \zeta_k \) it can be seen that basically \( \eta_k/\rho \) is independent of \( \rho \). This enables us to express the error bound in terms of \( \rho \). We summarize our results in the following theorem:

**Theorem.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( \Gamma_n, \rho \). Put \( \mu_k = (1 - \rho^2)/\lambda_k \), \( k = 1, \ldots, n \). Let \( \zeta_k \) and \( \varepsilon_k \) be corresponding quantities as in (3) and (4).

Define

\[ \eta_k = \min_{i \neq k} |\zeta_i - \zeta_k| \quad \text{and} \quad \varepsilon = \max_k |\varepsilon_k|. \]

Assume

\[ \eta := \min_k \eta_k > 2\varepsilon, \]

then

\[ |\mu_k - \zeta_k| \leq \min(\varepsilon_k, \varepsilon_k^2/(\eta_k - 2\varepsilon)), \quad k = 1, \ldots, n. \]

This implies in particular, provided \( \eta \gg \varepsilon \), that

(i) \( \mu_k = \zeta_k^- + O(\rho(1 - \rho)^2) \), \( \rho \uparrow 1 \),

(ii) \( \mu_k = \zeta_k + O(\rho^3) \), \( \rho \to 0 \),

(iii) \( \mu_k = \zeta_k^+ + O(\rho(1 + \rho)^2) \), \( \rho \downarrow -1 \),

where the constants involved can be effectively computed.

On the whole, one may expect the Rayleigh Quotients \( \zeta_k^{(\text{sign}(\rho))} \) to give good approximations of \( \mu_k \), especially when \( n \) is not too small. For small values of the parameter \( \rho \), the Rayleigh Quotient \( \zeta_k \) is rather precise; if \( \rho \) is close to 1, it is advisable to use \( \zeta_k^- \) and likewise one should take \( \zeta_k^+ \) in case \( \rho \) is close to -1.

It is not difficult, using the Propositions 1 and 2 and the Theorem, to work out exact error bounds for the approximations \( (1 - \rho^2)/\zeta_k \) of \( \lambda_k \). We leave the details to the interested reader.
References

Dickey, David A. and Wayne A. Fuller, 1979, Distribution of the estimators for autoregressive time series with a unit root, JASA 74, no. 366, 427–431.