

# On the shape of a violin

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## Abstract

For centuries luthiers—that is instrument makers of violins and other stringed instruments—had no more sophisticated tools at their disposal to define the shape of their instruments than straightedge and compass. Today modern aids are available in terms of computational power and expertise in graphic design to assist them in this respect. This raises the following question: how can these powerful computational techniques be applied in the process of searching for a form of the violin, both pleasing to the eye and optimal in some mathematical sense? In this paper I use parametric cubic splines in an attempt to come close to and possibly improve upon—strictly in a mathematical and visual sense—the shape of a violin as laid down by the great masters of the past. The main reasons for choosing the cubic spline are: good approximation properties, simplicity of construction, and most importantly, its unique curvature properties.

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## 1 Introduction

I learned to appreciate classical music early in life. My mother was a professional musician and I loved listening to her playing the piano for hours on end. At the age of eight I was given the choice between starting to learn playing the violin or the violoncello. I would have liked to play the piano, like my mother, but no, there was a far greater demand for stringed instruments she said, and so the choice was made for me: the cello was going to be my companion from then on. At first I hated my instrument, I hated having to go to lessons every Friday instead of playing with my friends, and I hated preparing for them under the watchful eye of my mother. But gradually I got the hang of it, and after a few years I began to like playing, and I also liked being invited to join the school orchestra and several chamber music groups. It was then that I started to be amazed at how the sound of a dead piece of wood could touch people so deeply. I promised myself then that, given half a chance, I would try to learn more about the miraculous mechanism that made this happen.

Well, as with many good intentions, this one got forgotten about when new exciting vistas opened up. I was doing well in mathematics, and by the time I went to university to study this fascinating discipline, my cello lessons stopped, and I played less and less until I ceased playing altogether. I became a mathematician and I spend the next forty years teaching and thinking about mathematics. Apart from a few rare instances when musical friends lured me back into making music again, this situation remained more or less unchanged until my retirement a few years ago. I then remembered the promise I made to myself and one day, when my youngest sister and I happened to pass the workshop of a violin maker, she insisted that I should ask inside about the possibility of learning more about the making of stringed instruments. I didn't believe that anyone could be interested in entertaining an old pensioner in need of killing a few hours, but my sister urged me on, and to make a long story short, the lady luthier of the shop was very helpful. The information she offered enabled me to enroll into a serious violin making course, offered by the Centrum voor muziekinstrumentenbouw (Cmb) at Puurs near Antwerp, building classical instruments in the tradition of the great masters of the past, and recently I finished my first instrument, a viola da gamba after Pieter Rombouts of Amsterdam (1705). In the past four years I learned much about the art of lutherie, but I have still a long way to go on my way to unraveling the secret of the magic sound produced by a well-played violin bearing the name of Antonio Stradivari.

I should come to the point. Thinking about violin making, it struck me early on that its shape has changed very little over the past hundreds of years. Apart from differences in size, coloring, and decoration, there are very few noticeable differences in shape: the form of a classical violin, viola or violoncello seems pretty much established. So where did this shape come from? And why did they stick to it so tenaciously? It is tempting to believe that the classical instruments of the great masters can hardly be improved, so the best advice is to copy them meticulously and hope for a similar sound. On the other hand, the shape of most other stringed instruments is not so strictly adhered to, that is to say, there is much more variety in the form that defines a guitar, a gamba or most other baroque

instruments. So, I wondered, what makes the shape of a member of the violin family so special? Studying the ways in which stringed instruments are constructed, I learned that, on the one hand it is common practice to copy well-known famous instruments in minute detail, on the other hand, if a new model is desired, with a few exceptions, only straightedge and compass are used in the construction. Often, the ways in which such constructions are laid out are complicated, and appear to be rather ad hoc (see [1, Tafel I]). “We need a round curve here, so which circle serves our purpose best”, seems to be the adage, and where circles meet, the sharp intersection is generally smoothed over. An exception is the catenary, a curve defined by the formula

$$y = a \cosh(x/a) = \frac{a}{2} (e^{x/a} + e^{-x/a}) \text{ with } a > 0,$$

which sometimes helps to shape the arching of the plates of the violin. The constant  $a$  can be expressed in terms of the length of the centerline of the backplate. But generally mathematical formulae are shunned.

The measurements required for the construction sometimes follow certain patterns, like the golden section geometric series

$$c\phi, c\phi^2 = c(1 + \phi), c\phi^3 = c(1 + 2\phi), \dots, c\phi^{k+1} = F_{k-1} + F_k\phi, \dots$$

where  $c > 0$  is a fixed measurement, like the length of the centerline of the backplate of the instrument,  $\phi = (1 + \sqrt{5})/2$  and  $F_k$  is the  $k$ -th Fibonacci number (see [4]). Often they seem to come from local considerations and follow no general rule or philosophy. Although it is



Figure 1: Inside template of the base model.

obvious that traditionally no other tools than straightedge and compass were available, it seems just as obvious that in the present day and age there is a variety of tools that could much better serve to find the ideal curve to satisfy the need for beauty and elegance of the

soundbox of a violin. Why stick to straight lines and circles where there is a wealth of other curves available? With the help of modern sophisticated software like computer algebra systems and graphic CAD software they are no longer beyond the practical realization. But what is ideal? Of course this is mainly a matter of taste. But I wonder, could there be a kind of ‘natural curving’ that would please most and could be backed up by some sort of rationale?

So let us analyze what is needed to construct the contour of a stringed instrument that belongs to the violin family. Without loss of generality we may take the violin. When starting on a new instrument it is common practice to first make a template that serves as a model for the mould to which the sides of the violin (the ribs) have to be glued (see figure 1). The ribs are approximately 1.2 mm thick and both front and back plates protrude from the sides by another 2.5 mm so that the outer form of the violin is slightly larger which is especially noticeable at the points where the upper and lower parts change into the C’s (see figure 2: rotating it counterclockwise over 90 degrees we distinguish the upper part and the wider lower part and the two middle C parts). Our aim is modest, we are merely



Figure 2: 2D backplate copy of the base model.

looking at the two-dimensional form of the soundbox of a violin—for this it is best to take the back—and try to find a natural way to construct it. When designing a new violin, it is the template that we have to construct first. However, since we wish to compare our construction to actual instruments, we shall instead focus on the outer form. Needless to say, we are not concerned with acoustics, however important that undoubtedly is, but with esthetics. Obviously our design must look like a violin, which places heavy restrictions on it. Incidentally, most violin makers will admit that, although professional musicians choose their instruments for the quality of its sound, when they have to make an ultimate choice, it is often the beauty of the form that wins.

It is obvious that starting the construction we have to set off with a number of given

measurements. These roughly determine the outer form of the instrument. Initially we have to decide on the length of the center line, which divides the back in two symmetric halves. We call this the height and it is such a symmetric half we are interested in. We also have to know the position and size of the C part, and the largest and smallest width on the upper and lower parts. All this is necessary so that the final result could rightly be called a violin. This also means that, although we may choose our own measurements, we must not lose sight of the fact that the margins are rather small.

Next we choose a number of points on the contour in agreement with the measurements we set out with. We shall refer to these points as guide points. There are a few obvious ones, like the endpoints of the center line and the endpoints of upper and lower parts and of the C part and maybe a few others, corresponding to maximal and minimal width for instance (see figure 5). Let us set them out on paper. Do we see the rudimentary shape of a violin? In other words, if we join the points chosen by a ‘most natural’ curved line—that is bending where needed but rather gradually—does the result resemble the shape of a violin? If not, we choose some more points, and so we finally arrive at a minimum but sufficient number of guide points for our symmetric half. We want to keep this number at a minimum to let the spline engine do its work. Finally we construct a continuous curve through the given points in such a way that our requirements of ‘natural curvature’ are met. This will be achieved by parametric cubic splines for each of its three parts. We also will consider the mathematical curvature at each point of the curve so that we can hopefully judge which circle provides the best fit at each point of the contour.

The important point I wish to make here is that we shall try to create the ‘ideal’ curve with as few guide points as possible, where ideal is meant in the sense of ‘visually pleasing’ with no unnecessary curving. Observe that the points taken from existing instruments are probably cursed with small errors, and therefore, having groups of points close together often goes with quite a bit of erratic and thus unnecessary bending or oscillation (overshoot), whereas with few points spaced relatively wide apart this phenomenon does not occur so easily and the spline’s curving follows a more natural path. Also, it will turn out that, with the guide points chosen in accordance with the measurements of the base model, an instrument bearing the name of Antonio Stradivari and the year 1689—see figure 2 for a copy in the process of being constructed—we get contours that compare very favorably with those of the actual instrument. But let us have a look at cubic splines first.

## 2 Cubic splines

Suppose we have chosen the guide points. The graph of the contour function should run through these points, so it makes sense to consider interpolation. Polynomial interpolation seems for our purpose a natural choice because by a famous theorem of Weierstraß a function continuous on the closed interval  $[a, b]$  can be approximated on that interval by polynomials to any given precision. Because of the relatively large number of guide points we get an interpolation polynomial of high degree. Now high degree polynomial interpolation is generally not a good idea, a major drawback being its oscillatory behavior, causing

unnecessary bending. An alternative is piecewise polynomial interpolation of low degree. But we also want the resulting curve to be smooth at the knots, that is at the points where neighboring polynomial pieces meet. Therefore piecewise linear polynomial interpolation is not good enough, we need at least piecewise quadratic polynomial interpolation. Polynomial interpolation with an interpolation condition on the derivatives as well is known as Hermite interpolation. But piecewise quadratic Hermite interpolation does not give us enough freedom to control the smoothness at the knots as we shall see shortly. Thus the best choice for our problem appears to be piecewise cubic interpolation. Let us see what this means.

Let  $\{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$  be a data set with  $a = x_0 < x_1 < \dots < x_n = b$ . A piecewise cubic interpolation polynomial on the given data set passes through the points of this set, but it is quite obviously not unique. Therefore we can impose some smoothness conditions at the knots  $x_0, x_1, \dots, x_n$ . A piecewise cubic Hermite interpolation polynomial has a continuous derivative and a cubic spline is a piecewise cubic interpolation polynomial with a continuous second order derivative. Although this is not customary, I shall often refer to a piecewise cubic Hermite interpolation polynomial as a cubic subspline.

Let  $s$  be a cubic subspline or a cubic spline on the interval  $[a, b]$ . Then  $s$  satisfies the conditions:

1.  $s = s_i$  is a cubic polynomial on the interval  $[x_{i-1}, x_i]$  for each  $i = 1, \dots, n$ ,
2.  $s(x_i) = y_i$  for  $i = 0, \dots, n$ ,
3.  $s_i^{(j)}(x_i) = s_{i+1}^{(j)}(x_i)$  for  $i = 1, \dots, n-1$  and  $j = 0, 1, \dots, m$ , where  $m = 1$  in case of a subspline and  $m = 2$  in case of a spline.

Condition 1 tells us that  $s$  is a piecewise cubic polynomial and condition 2 says that  $s$  is an interpolation function on the given data set. The third condition expresses the smoothness of  $s$ : it says  $s \in C^m[a, b]$ , which means that  $s$  is an  $m$  times continuously differentiable function on the closed interval  $[a, b]$ . The  $s_i$  are cubic polynomials and therefore  $s$  can be explicitly given by  $4n$  coefficients. On the other hand the interpolation and smoothness conditions amount to a total of  $n+1+(n-1)(m+1)$  equations. As  $n+1+(n-1)(m+1) \leq n+1+3(n-1) = 4n-2$ , extra conditions are needed to explicitly determine  $s$ . A piecewise *quadratic* Hermite interpolation polynomial would be determined by  $3n$  coefficients, and the interpolation and smoothness conditions would give a total of  $3n-1$  equations. This leaves only one degree of freedom, where a cubic subspline under the same conditions leaves  $n+1$  degrees of freedom, so that we have an extra degree of freedom at every knot. The two extra degrees of freedom in case of a cubic spline are commonly used up at the endpoints.

Cubic splines are by far the most popular of all spline functions, they are particularly useful for approximation purposes. In particular, every function continuous on a closed interval  $[a, b]$  can be arbitrary well approximated on  $[a, b]$  by cubic splines provided sufficiently many knots are available. There are very many types of spline functions depending on the choices of the degree and the order of differentiation required as well as the extra conditions. General information on spline functions can be found in [3] and [15].

The definition above does not give sufficient information to uniquely determine the cubic subspline function. But if we happen to know the values of the derivatives at the knots we would have  $n + 1$  extra conditions, which would add up to exactly as many conditions as equations. So let us first assume that the values of the derivative of  $s$  at the knots are known in one way or another. Write  $d_i = s'(x_i)$  for  $i = 0, 1, \dots, n$ . As  $s_i$  is a cubic polynomial depending on the four values  $y_i, y_{i-1}, d_i$  and  $d_{i-1}$  we may write  $s_i(x)$  as a linear combination of these values with cubic polynomials in  $x$  as coefficients, i.e.

$$s_i(x) = A_i(x)y_i + B_i(x)y_{i-1} + C_i(x)d_i + D_i(x)d_{i-1} \quad \text{for } i = 1, \dots, n. \quad (1)$$

Applying the four interpolation conditions

$$s_i(x_i) = y_i, \quad s_i(x_{i-1}) = y_{i-1}, \quad s'_i(x_i) = d_i, \quad s'_i(x_{i-1}) = d_{i-1}$$

leads to the sixteen equations

$$\begin{aligned} A_i(x_i) &= 1, & B_i(x_i) &= C_i(x_i) = D_i(x_i) = 0, \\ B_i(x_{i-1}) &= 1, & A_i(x_{i-1}) &= C_i(x_{i-1}) = D_i(x_{i-1}) = 0, \\ C'_i(x_i) &= 1, & A'_i(x_i) &= B'_i(x_i) = D'_i(x_i) = 0, \\ D'_i(x_{i-1}) &= 1, & A'_i(x_{i-1}) &= B'_i(x_{i-1}) = C'_i(x_{i-1}) = 0. \end{aligned}$$

These sixteen equations involve  $4 \times 4 = 16$  unknowns, namely the coefficients of the cubics  $A, B, C$  and  $D$ . This uniquely determines  $s_i$ , and as this can be done for every  $i$ , hence also  $s$ . Write  $h_i = x_i - x_{i-1}$  for  $i = 1, \dots, n$ . Then

$$\begin{aligned} A_i(x) &= \frac{3h_i(x - x_{i-1})^2 - 2(x - x_{i-1})^3}{h_i^3}, & B_i(x) &= \frac{h_i^3 - 3h_i(x - x_{i-1})^2 + 2(x - x_{i-1})^3}{h_i^3}, \\ C_i(x) &= \frac{(x - x_{i-1})^2(x - x_{i-1} - h_i)}{h_i^2}, & D_i(x) &= \frac{(x - x_{i-1})(x - x_{i-1} - h_i)^2}{h_i^2}, \end{aligned}$$

as is easily verified. So if  $d_0, d_1, \dots, d_n$  are given we are done. As we shall see later, we can choose our guide points in such a way that this is indeed the case. However a cubic subspline has a big drawback in that its second order derivative may be discontinuous and this implies discontinuous curvature—see (4) for the definition of curvature and see page 15 for the section on curvature. Moreover large jumps in curvature can usually be spotted by the naked eye as less appealing. In figure 3 we compare the graphs of the cubic spline and cubic subspline through the points  $(0, 1.8)$ ,  $(1, 2)$ , and  $(3, 1)$ . It is quite clear that the ‘hump’ in the graph of the cubic subspline is not as natural and ‘pleasing to the eye’ as the corresponding one in the graph of the spline.

So, what else can we do? How should we choose these  $d$ -values? There are many ways in which we can do this (see [8, 4.8] and [10, 3.4]). In view of our ultimate goal, we would like to choose them in a ‘visually pleasing’ way by which we mean a kind of natural smoothness and an avoidance of unnecessary bending. In [10, 3.4] a description is given of a method that avoids overshooting, which is the main cause of oscillation. Although this

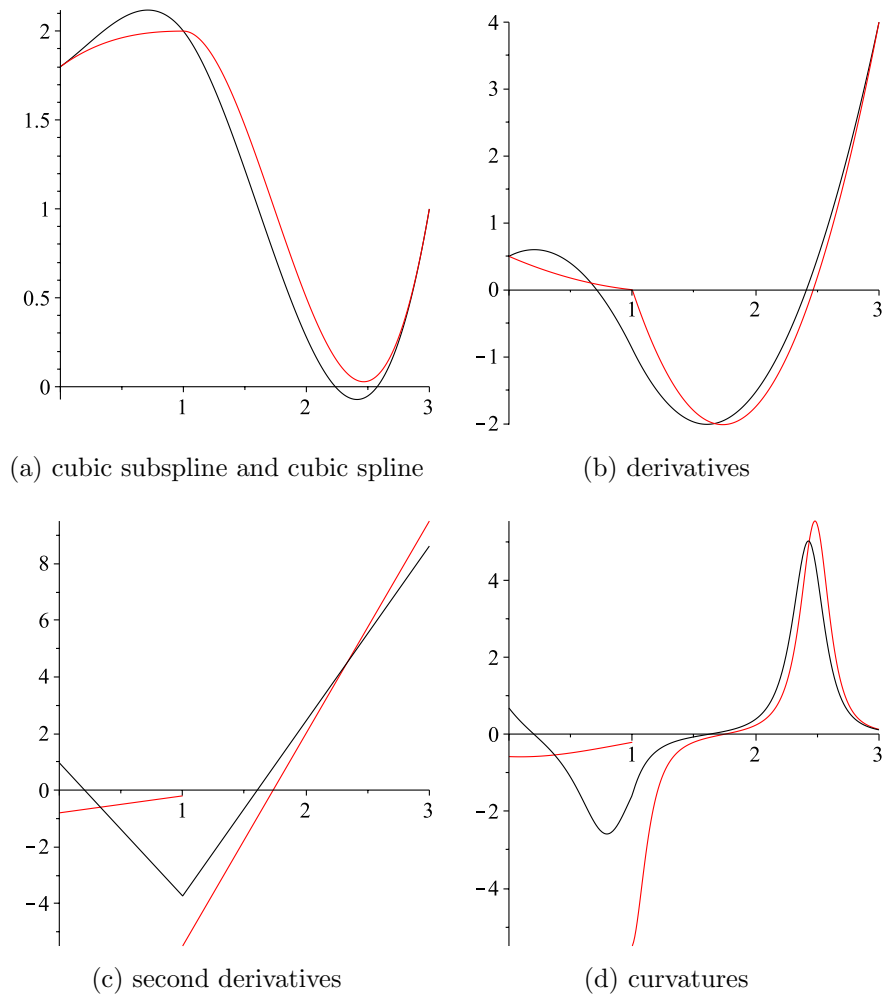


Figure 3: Spline (black) and subspline (red) through the points  $(0, 1.8)$ ,  $(1, 2)$ , and  $(3, 1)$ .

approach has certain advantages, we prefer a different one mainly because of more natural curvature properties as we shall see shortly.

Another way to deal with the  $d_i$  is to force extra conditions on  $s(x)$  and this brings us to the cubic spline. As we saw before, in this case we have  $4n$  unknown coefficients and  $4n - 2$  equations. Therefore we may impose two extra boundary conditions. A natural choice is  $s''(a) = s''(b) = 0$ , which gives the so-called *natural cubic spline*. Another choice is  $(s'(a), s'(b)) = \alpha$  for a given vector  $\alpha = (\alpha_1, \alpha_2)$ , and this is known as the *clamped cubic spline*.

The existence of cubic splines can be shown in a constructive way. The function  $s''$  is a piecewise linear polynomial. Write  $\sigma_i = s''(x_i)$  for  $i = 0, 1, \dots, n$ . Then for  $i = 1, \dots, n - 1$  we have

$$s''(x) = \begin{cases} \frac{x-x_i}{x_{i+1}-x_i}\sigma_{i+1} + \frac{x_{i+1}-x}{x_{i+1}-x_i}\sigma_i & \text{for } x_i \leq x \leq x_{i+1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}\sigma_i + \frac{x_i-x}{x_i-x_{i-1}}\sigma_{i-1} & \text{for } x_{i-1} \leq x \leq x_i \end{cases},$$

because  $s''$  is a linear function. Working out the repeated integrals

$$\int_{x_i}^{x_{i+1}} \int_{x_i}^x s''(t) dt dx \quad \text{and} \quad \int_{x_{i-1}}^{x_i} \int_x^{x_i} s''(t) dt dx$$

in two ways and eliminating  $s'(x_i)$  from the resulting equations eventually leads to the relations

$$\omega_i \sigma_{i-1} + 2\sigma_i + (1 - \omega_i) \sigma_{i+1} = r_i \quad \text{for } i = 1, \dots, n - 1, \tag{2}$$

where

$$\omega_i := \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad \text{and} \quad r_i := \frac{6}{x_{i+1} - x_{i-1}} \left( \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right).$$

Combining relations (2) into a single matrix equation, we get for the natural boundary conditions, the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \omega_1 & 2 & 1 - \omega_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \omega_2 & 2 & 1 - \omega_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 2 & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & & 0 & \\ 0 & 0 & 0 & 0 & \cdots & \omega_{n-1} & 2 & 1 - \omega_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} = \begin{pmatrix} 0 \\ r_1 \\ \vdots \\ r_{n-1} \\ 0 \end{pmatrix}.$$

The coefficient matrix is tridiagonal and strictly diagonally dominant, because  $0 < \omega_i < 1$  for each  $i$ . This means that this system is uniquely solvable (see [6, Theorem 6.1.10 on page 349]) which implies that the natural cubic spline exists and is unique. It also gives us the values of  $s''(x)$  at the knots. Tracing back down the intermediate relations gives us the coefficients of each  $s_i$ . In case of the clamped cubic spline we find a slightly more complicated matrix equation, but the conclusions remain the same.

For both sets of boundary conditions and for any function  $f \in C^m[a, b]$  with  $m \geq 2$  and  $(f'(a), f'(b)) = \alpha$  we can show the inequality

$$\int_a^b [f''(x)]^2 dx \geq \int_a^b [s''(x)]^2 dx. \tag{3}$$

The proof runs as follows. Consider

$$\int_a^b [f''(x) - s''(x)]^2 dx + \int_a^b [s''(x)]^2 dx = \int_a^b [f''(x)]^2 dx - 2 \int_a^b s''(x)[f''(x) - s''(x)] dx.$$

Using integration by parts on the last integral on the right gives

$$\int_a^b s''(x)[f''(x) - s''(x)] dx = s''(x)[f'(x) - s'(x)] \Big|_{x_0}^{x_n} - \int_a^b s'''(x)[f'(x) - s'(x)] dx.$$

The first term on the right vanishes because of the boundary conditions and so does the second term in view of the fact that  $s'''$  is constant on each subinterval  $(x_{i-1}, x_i)$ . Indeed, if  $s'''(x) = c_i$  for  $x \in (x_{i-1}, x_i)$  and  $i = 1, \dots, n$ , then

$$\int_a^b s'''(x)[f'(x) - s'(x)] dx = \sum_{i=1}^n c_i \int_a^b [f'(x) - s'(x)] dx = 0.$$

Inequality (3) has an important geometric interpretation that partly explains the reason for the popularity of the cubic spline. The mathematical curvature  $\kappa(x)$  of a twice continuously differentiable function  $f : [a, b] \rightarrow \mathbb{R}$  at the point  $x \in [a, b]$  is defined by the formula

$$\kappa(x) = \frac{f''(x)}{(1 + [f'(x)]^2)^{\frac{3}{2}}}. \quad (4)$$

The curvature of a circular arc of radius  $R$  is  $1/R$  or  $-1/R$ , depending on the orientation of the circle. Assuming  $|f'(x)| \ll 1$  on  $[a, b]$ —admittedly, this is not always the case—we see that the norm  $\|\kappa\|_2^2$  is approximately equal to  $\int_a^b [f''(x)]^2 dx$  so that inequality (3) now says that of all the  $C^m[a, b]$  functions with  $m \geq 2$  and satisfying the interpolation conditions, the cubic spline has the smallest total curvature in the sense of the  $\ell_2$ -norm. Of course,  $\int_a^b [f''(x)]^2 dx$  merely gives a coarse measure of the total curvature.

There is yet another interpretation of inequality (3) and this one explains the reason for the name ‘spline’ that is given to this interpolation function. Consider a thin homogeneous isotropic flexible rod whose center line is given by a function  $f : [a, b] \rightarrow \mathbb{R}$ . Then the total bending energy is given by the formula

$$C \int_a^b \frac{[f''(x)]^2}{(1 + [f'(x)]^2)^3} dx \approx C \int_a^b [f''(x)]^2 dx$$

for a certain constant  $C$  and assuming  $|f'(x)| \ll 1$  on  $[a, b]$ . If such a rod is forced to go through a number of fixed points, in such a way that only forces perpendicular to the rod are applied, it assumes a position of minimal energy. Therefore, inequality (3) now asserts that the center line of this rod approximately follows the natural cubic spline through these points. Outside of the interval  $[a, b]$ , no force is applied to the rod, and therefore it assumes the natural shape of a straight line. In that sense the boundary conditions  $s''(a) = s''(b) = 0$  should be seen as ‘natural’. It now makes sense why this type of interpolation function was given the name ‘spline’, because a mechanical spline is a thin flexible rod that is used by draughtsmen (e.g. in shipbuilding) for drawing smooth curves through a number of fixed points. It was I.J. Schoenberg who introduced the name ‘spline’ in 1946 (see [13]). See also the foreword by A. Robin Forrest in [2].

Cubic splines have very good approximation properties. In fact, if  $s$  is a cubic spline that interpolates  $f \in C^m[a, b]$  at the points  $a = x_0 < x_1 < \dots < x_n = b$  then

$$\|s - f\|_\infty = O(h^{m+1}) \text{ as } h \downarrow 0, \text{ where } m = 1, 2, 3 \text{ and } h := \max_{i=1, \dots, n-1} |x_i - x_{i-1}|.$$

See [5] and [17, 9.7.3].

Now the curves we need for our purpose cannot in all cases be given by the graphs of spline *functions*, we need *parametric* splines. We understand a parametric spline to be a plane curve given by

$$\{(x(t), y(t)) : a \leq t \leq b\},$$

where  $x$  and  $y$  are spline functions of the parameter  $t$ . Often one chooses  $a = 0$  and  $b = 1$ . How should the parametrization be chosen? This is an important point. The interpolation points are generally not uniformly spaced, and therefore different parametrizations should give different splines. Also, the curves we are after are certainly non-singular, so our parametrized curves should also be non-singular. The most natural choice is to take  $t$  to be the arc length of the curve. But it is almost always rather difficult to find an explicit expression for the arc length of a given curve, and our curves are no exception. We should

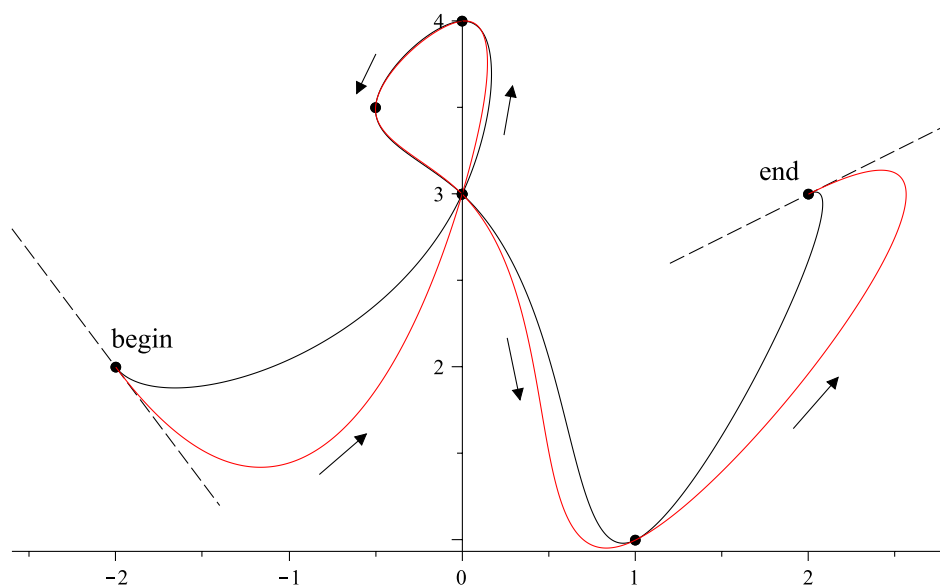


Figure 4: Parametric splines through the same points and with equal end conditions but with different multiplication factors at the endpoints.

also take into account the fact that successive guide points are not placed at equal distances. So, if for  $i = 0, 1, \dots, n$  the points  $A_i = (x_i, y_i)$  are the guide points, then we choose  $t_0 = 0$ ,  $s_{i+1} = s_i + \|(x_{i+1} - x_i, y_{i+1} - y_i)\|_2$  for  $i = 0, 1, \dots, n - 1$  and  $t_i = s_i/s_n$  for  $i = 1, \dots, n$ . Then  $t_n = 1$  and  $t_{i+1} - t_i$  is a linear approximation of the relative arc length between the

guide points  $A_i$  and  $A_{i+1}$ . Dealing with clamped cubic spline functions, we need to choose the tangent values at both ends to make the spline unique. With parametric clamped cubic splines we have even more freedom. Indeed, let the curve be given by  $f(x, y) = 0$  with a parametrization as given above. Traversing the curve from  $t = t_0$ , let us set off in the direction of the vector  $\alpha$  with  $\|\alpha\|_2 = 1$ . Then  $x'(t_0) \neq 0$  or  $y'(t_0) \neq 0$ . Without loss of generality we assume  $x'(t_0) \neq 0$ . It then follows, that in a neighborhood of  $A_0$  at  $t_0$ , the curve can be given by  $y = \phi(x)$ . Now let  $x'(t_0) = m \cdot \alpha_1$  and  $y'(t_0) = m \cdot \alpha_2$ , then  $\phi'(t_0) = y'(t_0)/x'(t_0) = \alpha_2/\alpha_1$ , and the multiplication factor  $m > 0$  drops out.

So the value we choose for  $m$  does not effect the direction of the tangent at  $t_0$ . Changing the multiplication factor does not alter the tangent, but the larger  $m$  the closer the curve is drawn towards the tangent. Naturally this also applies to the other endpoint  $A_n$  at  $t_n$ . We shall always take the multiplication factor positive. We therefore may have to alter the sign of the direction of the tangent, depending on the way we traverse the curve. In figure 4 the multiplication factor of the red spline is much larger than that of the black spline. The directions of the tangents (the dashed lines) at the endpoints are  $[3, -4]$  and  $[-2, -1]$ , respectively.

### 3 Choosing guide points

Choosing the number of guide points and the points themselves is crucial. First, we should not choose too many points for fear of designing the contour ourselves, instead of leaving this to the spline engine. Further, the choice of guide points should be mainly imposed by necessity, because of the obvious requirements of size and shape. To start with, we have to decide on the size of the instrument. So let us consider a standard violin. The

| Standard violin | mm  |
|-----------------|-----|
| length          | 356 |
| upper width     | 168 |
| middle width    | 112 |
| lower width     | 208 |

Table 1: Back plate measurements.

measurements vary only slightly with a variation of the length of the body of at most 10 mm. It is easiest to work with an existing model; we haven chosen a model used in the violin class of the Cmb (see page 2), which comes from a violin constructed by A. Stradivari in 1689. We shall refer to it as the base model. Some of the measurements of this model are given in Table 1. A list of 113 points from which the outer form of this model (see figure 5) can be constructed is available on my homepage. Useful information on measurements is given in [16].

The procedure is now as follows. We consider the backplate of our model, or rather half of it, and we try to find points on its contour at the most significant positions. Nineteen

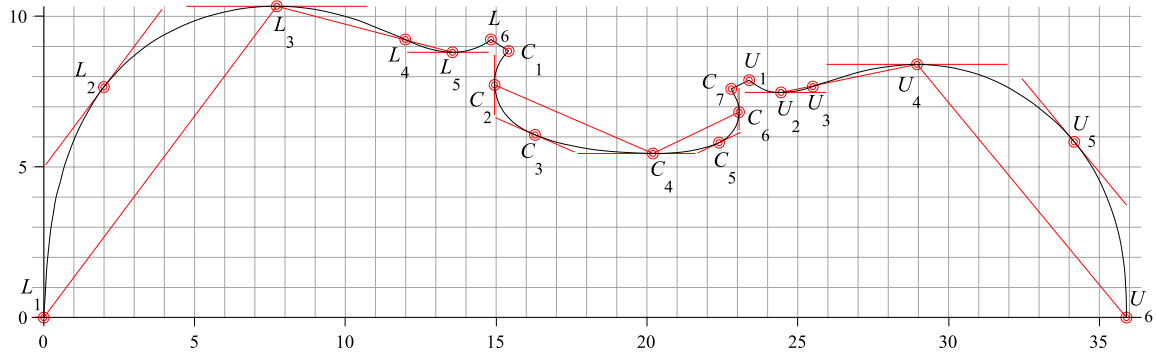


Figure 5: Significant points on the base model’s outer form with construction lines.

possible candidates with an indication on their construction are shown in figure 5. We see several points in extremal positions ( $L_3$ ,  $L_5$ ,  $C_2$ ,  $C_4$ ,  $C_6$ ,  $U_2$ , and  $U_4$ ), there are points with tangents parallel to line segments  $L_1L_3$ ,  $C_2C_4$ ,  $C_4C_6$ , and  $U_4U_6$  ( $L_2$ ,  $C_3$ ,  $C_5$ , and  $U_5$ ), and points at the intersection of line segments  $L_3L_5$  and  $U_2U_4$  and the base model ( $L_4$  and  $U_3$ ). From this set of points we shall choose a subset, the guide points. And finally, these guide points will serve as interpolation points for our parametric cubic splines. We shall also compare the result with the base model.

Now half the contour can be naturally split into three separate parts, the L(ower), the middle or C and the U(pper) parts, see figure 5. For each of these three parts we shall construct a parametric cubic spline. First we need to decide on the position of the endpoints of these three parts, namely  $L_1$  and  $L_6$ ,  $C_1$  and  $C_7$ ,  $U_1$  and  $U_6$ . There cannot be

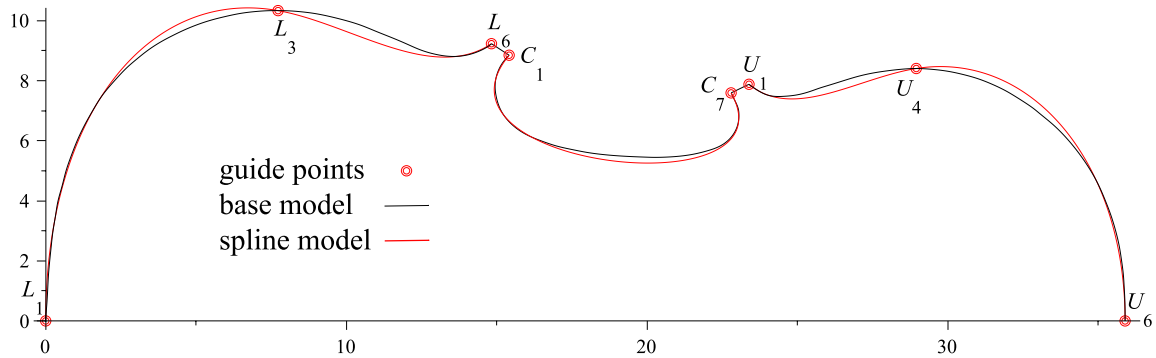


Figure 6: 8-point spline compared with the outer form of the base model.

much doubt about the necessity of the choice of these points as guide points, but clearly this choice cannot possibly be sufficient; at least two more guide points are needed to get anything like the correct shape, typical for a violin. We also must choose the direction of the tangents at these six points and their multiplication factors. An important point is this. We can choose the multiplication factors to suit us best, which could either mean so that we like the resulting curve best, or that it matches the contour of our base model best. The tangents at  $L_1$  and  $U_6$  must be vertical and the tangents at the other four points can be freely chosen, say to run at an angle anywhere between 30 to 60 degrees in a positive or negative sense. The upper and lower parts are quite similar, so let us consider

| Guide point | direction tangent | multiplication factor |
|-------------|-------------------|-----------------------|
| $L_1$       | $[0, 1]$          | 32                    |
| $L_6$       | $[1.5, 1]$        | 20                    |
| $C_1$       | $[-1, -1.2]$      | 14.6                  |
| $C_7$       | $[-1, 1.8]$       | 13.5                  |
| $U_1$       | $[1.2, -1]$       | 17                    |
| $U_6$       | $[0, -1]$         | 24                    |

Table 2: Tangent directions at end points in figure 6.

the lower part. Provided we choose a point between  $L_1$  and  $L_6$  with an ordinate value larger than that of  $L_6$ , the resulting parametric spline through these three points will have the right sort of shape. This is also true for the upper part, the C part however does not need another point. The extra points that certainly satisfy the restrictions are the highest points on the lower and upper parts, namely  $L_3$ , and  $U_4$ . In figure 6 we compare the shape generated by the cubic parametric splines through these 8 guide points with that of the base model. See also table 2 for the tangents and the multiplication factors. The latter

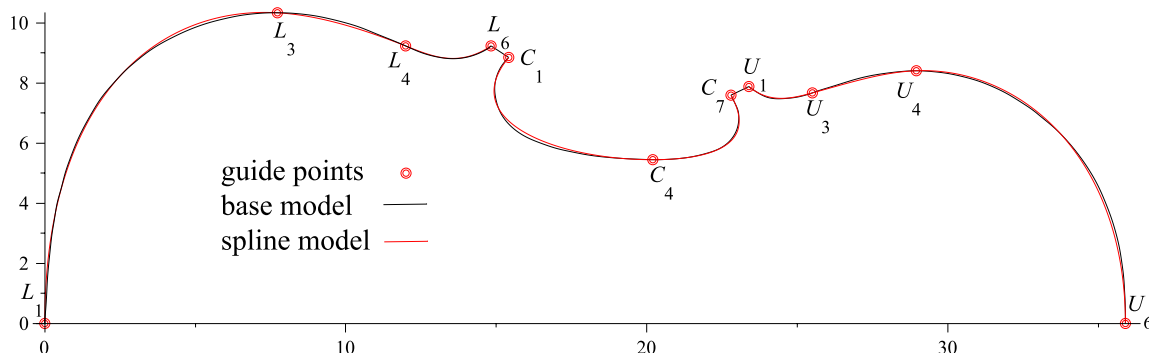


Figure 7: 11-point spline compared with the outer form of the base model.

are chosen such that the spline curves at the end points optimally agree with those of the

base model. The directions of the tangents are read from the base model in the following way: large multiplication factors draw the curve towards the tangent, making the latter clearly visible if the multiplication factor is large enough. Although the shape of the spline curve in figure 6 seems quite acceptable, there is also room for improvement. Observe that the points  $L_3$  and  $U_4$  are not at extremal positions on the spline, and that the C part spline curve possibly makes the waist too small. So if one wishes to stay closer to the base model, extra points are needed, like  $L_4$ ,  $C_4$ , and  $U_3$ . The points  $L_4$  and  $U_3$  have the effect of ‘flattening off’ the splines towards the base model. Of course, any points between  $L_3$  and  $L_5$ , and between  $U_2$  and  $U_4$  will have that effect. In figure 7 and table 3 the results are given. Observe that the directions of the tangents are the same as in table 2, but the multiplication factors are not. Although not perfect, the match of the 11-point spline is

| Guide point | direction tangent | multiplication factor |
|-------------|-------------------|-----------------------|
| $L_1$       | $[0, 1]$          | 34.5                  |
| $L_6$       | $[1.5, 1]$        | 28                    |
| $C_1$       | $[-1, -1.2]$      | 13.1                  |
| $C_7$       | $[-1, 1.8]$       | 16.2                  |
| $U_1$       | $[1.2, -1]$       | 21                    |
| $U_6$       | $[0, -1]$         | 25                    |

Table 3: Tangent directions at end points in figure 7.

quite good. Other choices of guide points can be made, and a perfect match can be easily obtained by taking a few extra points from the the 19 significant points of figure 5.

## 4 Curvature

We have seen that the combined spline contours through the 11 given guide points provide a reasonably close fit for the base model. Another interesting point that can be made is about the curvature: how does the curvature of the base contour change from point to point? Can the answer throw light on the way luthiers in their straightedge and compass constructions use circular arcs?

The curvature of a twice continuously differentiable function at each point of its graph has been given in section 2, see equation (4). From this the formula for the curvature of a parametric curve  $C = \{(x(t), y(t)) : a \leq t \leq b\}$  at each value of the parameter  $t$  can be deduced easily:

$$\kappa(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{\frac{3}{2}}}. \quad (5)$$

For convenience we have dropped the  $t$  in the right-hand side of formula (5). In order to check this formula, suppose  $x'(t) \neq 0$  at the parameter value  $t = t_0$ . Then the curve  $C$  may be given in a neighborhood of  $t_0$  by an equation  $y(t) = \phi(x(t))$  for a twice continuously

differentiable function  $\phi$ . Then dropping the  $t$  again,  $y' = \phi'(x)x'$  and  $y'' = \phi''(x)(x')^2 + \phi'(x)x''$ , from which in combination with (4) the assertion easily follows. The curves we consider have no singular points. So if  $x'(t) = 0$  at  $t = t_0$  then certainly  $y'(t_0) \neq 0$ , and a similar argument may be given. Recall that the circle of curvature at point  $t = t_0$  has radius  $1/\kappa(t_0)$ . It is the circle whose center lies on the normal to the curve at that point and whose curvature agrees with that of the curve at  $t_0$ .

It follows from formula (5) that the curvature function  $\kappa(t)$  is continuous on the entire range  $[t_0, \dots, t_n]$ . Nevertheless, the curvature of  $C^2$ -curves is rather sensitive to small changes. If our curve is a parametric cubic spline, it is unlikely that  $\kappa(t)$  is differentiable at the knots. So our function  $\kappa(t)$  will probably not have a very smooth appearance. The

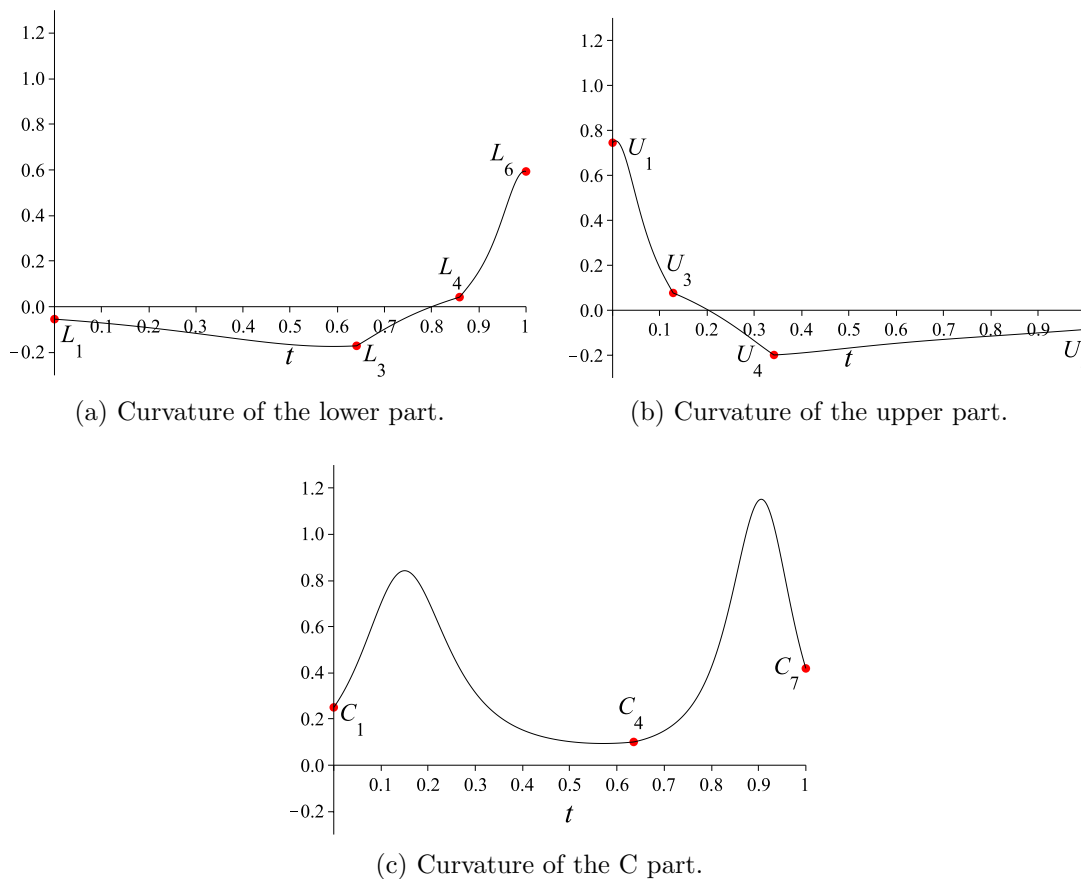


Figure 8: Curvature of the parametric 11-point spline curves.

contour of the Stradivari base model is obtained by careful observation of 113 successive data points, about 5 mm apart (see page 12). Joining these points by means of parametric cubic splines to obtain a close approximation of the contour gives a good result, but the corresponding curvature function does not look so good as a result of the phenomenon mentioned above.

Fortunately, the 11-point spline approximation of the base contour is also quite good,

and what is more, its curvature function is much smoother. It is therefore preferable to consider the curvature function of our parametric spline approximation. In figure 8 the curvature functions of the lower, C and upper parts are shown. It appears that the curvature at both ends  $L_1$  and  $U_6$  is rather constant and small. Although one should be very careful with one's interpretation of this, it could mean that the contour at both  $L_1$  and  $L_6$  is similar to a circular arc. Table 4 shows the approximate radius of the curvature circle at each of the guide points. We also observe in figure 8c that the curvature of the C part is rather regular; around the center the curvature is almost constant, which might indicate that the major curve resembles a circle with radius of approximately 9.9 cm. Also the small circular arcs near the endpoints are clearly visible.

| Guide point | $x(t)$     | $y(t)$     | $t$          | $\kappa(t)$   | $R$         |
|-------------|------------|------------|--------------|---------------|-------------|
| $L_1$       | 0.0        | 0.0        | 0            | -0.0538253196 | 18.57861703 |
| $L_3$       | 7.7279227  | 10.3352606 | 0.6406923777 | -0.1707587319 | 5.85621590  |
| $L_4$       | 11.9861762 | 9.2333787  | 0.8590647440 | 0.0425922197  | 23.47846643 |
| $L_6$       | 14.8249271 | 9.2336330  | 1            | 0.5932583552  | 1.68560627  |
| $C_1$       | 15.4128687 | 8.8493857  | 0            | 0.2506912160  | 3.98897104  |
| $C_4$       | 20.1975232 | 5.4481232  | 0.6353508310 | 0.1009567893  | 9.90522784  |
| $C_7$       | 22.7944348 | 7.5946695  | 1            | 0.4193279880  | 2.38476808  |
| $U_1$       | 23.3900054 | 7.8817742  | 0            | 0.7454555014  | 1.34146169  |
| $U_3$       | 25.4994239 | 7.6701966  | 0.1280534854 | 0.0777210168  | 12.86653264 |
| $U_4$       | 28.9525636 | 8.4071580  | 0.3413289745 | -0.1978771462 | 5.05364070  |
| $U_6$       | 35.8974966 | 0.0        | 1            | -0.0845792332 | 11.82323322 |

Table 4: Coordinates  $(x, y)$  with parameter value  $t \in [0, 1]$  of the 11 guide points (see figure 7) with their curvature  $\kappa(t)$  and radius  $R$  of the curvature circle. Coordinates and radius are measured in cm's.

## 5 Note on the practical application

So far we have considered the mathematical definition and properties of parametric cubic splines. But how can these be handled in practice? All calculations done so far, including all graphs, have been created by means of the Computer Algebra package Maple (see [9]). But in order to use splines one does not need intimate knowledge of a computational nature. It is sufficient to know how to work with a CAD system, like VectorWorks, TurboCad or AutoCAD, to name but a few. These systems use vector graphics and they implement parametric cubic splines through B-splines (B stands for Basis). These systems are mainly used for experimental design purposes, where curves are designed and used interactively. Now cubic splines—at least in the way presented here—are less useful for experimental purposes, because changing even a single interpolation point changes the entire spline instead of the parts closest to the point changed, so that all calculations have to be done

all over again. B-splines cure that problem by constructing the spline locally between two successive interpolation points as a linear combination of (four, in case of cubic splines) basis splines. Further, in design purposes there are generally no severe restrictions as to the points the curve has to pass through. With B-splines one uses control points to change the shape of the curve, and the curve generally does not pass through these control points. As we have no need for the extra flexibility of B-splines, we need not go into details here. For those who wish to know more about the mathematical background of B-splines, Bézier curves and NURBS (Non Uniform Rational B-Splines) we refer to [14] for a very nice and detailed overview. See also [11].

## 6 Conclusion

We have seen that by choosing guide points in a certain way a quite acceptable model for the backplate of a violin can be constructed with the nice curvature properties of cubic splines. Of course, the great masters would have had no need for these fancy methods—if they had known them—in their quest for the most gracious and beautiful shape, but we, ordinary mortals, might find the help modern methods can give us quite useful. These modern methods are easily accessible through CAD software. Even so, and fortunately too, in the search for suitable guide points, human intervention—in other words, the eye of the master—remains indispensable.

## 7 Acknowledgement

I am grateful to Sholem van Collem, professional violinist and accomplished amateur luthier, who put me on the right track in my search for a suitable set of guide points. I also thank him for his assistance in obtaining precise measurements for the outer form of the base model and above all for many hours of in-depth discussion on the making of violins.

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